Mathematical Facts and Processes Prerequisite to the Study of the Calculus

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CONTENTS

TABLES

CHAPTER I

INTRODUCTION

STATEMENT OF THE PROBLEM

THE purpose of this study is to ascertain to what extent the content of the courses of secondary mathematics is used in the solution of the problems of the calculus.

To list all the mental factors involved in the solution of even the simplest mathematical problem would be the task of the psychologist rather than the mathematician. In this study the writer endeavors to enumerate the mathematical facts, principles, formulas, and processes of secondary mathematics that are used by the student in the solution of the problems of the calculus, and to compare his findings with the requirements set forth in the syllabus for secondary mathematics of the College Entrance Examination Board. Analysis of the elements involved seems worth while in view of the many students who pursue the study of the calculus in college. Some persons may think that since only a small part of the ever increasing number of high school graduates go to college, and of this small group only a small percentage will study the calculus, this problem does not vitally affect the field of secondary mathematics. Even assuming that this is true, and also assuming that the majority of those who are preparing for college will study mathematics throughout the senior high school, is it not desirable to know how much and to what extent this knowledge of secondary mathematics is needed in the study of the calculus?

BASIS FOR THIS STUDY

The calculus was chosen as the basis for this study because it seems to be the goal to which secondary mathematics leads; because it is the foundation for all subsequent work in mathematics; because its fundamental principles are the most scientific and powerful tool of the modern mathematician; and because it still remains, almost exclusively, a college subject, although there is a

tendency to introduce the elementary notions of the calculus into the high school curriculum.

The courses that usually precede the study of the calculus are elementary, intermediate, and advanced algebra, plane and solid geometry, plane trigonometry, and analytic geometry. The term secondary mathematics is here used to include all the above listed courses, except analytic geometry. This investigation concerns primarily the uses made of secondary mathematics in studying the calculus, though consideration is also given to all applications of analytic geometry.

It is not assumed that all students who enter college have had all the courses mentioned above while in high school, for many students enter college with little preparation in mathematics beyond intermediate algebra and plane geometry. For this reason there are college courses in mathematics that are of lower rank than the calculus which are preparatory for it. The usual college courses which are prerequisite to the study of the calculus are college algebra, trigonometry, and analytic geometry. Since advanced algebra and trigonometry are taught in many high schools, and since they include about the same subject matter as the college courses in college algebra and trigonometry, these two courses are classed in the report as of secondary rather than collegiate rank, while analytic geometry, which usually follows the study of college algebra and trigonometry, will be considered apart from secondary mathematics, even though in some high schools the fundamental notions of analytic geometry are given.

In the solution of some of the problems in the calculus it may be necessary to use certain facts and skills that are not included in the courses in secondary mathematics. Some students may have to assume the needed formula or operation to be true, while others, with a thorough knowledge of secondary mathematics, can easily verify it. A well-trained student will have at his command the whole range of high school mathematics and will be able to discover how to deal with certain new topics that present themselves as the study of the calculus progresses.

Because a student has had the necessary amount of mathematics to enable him to do the calculus, it is not assumed that he will do it successfully, for the calculus has much that is its own, and only by being prepared and then applying himself to the task can the student discover all the beauties of this subject. It is not to be

understood that the calculus is a compilation of secondary mathematics. It is unique in its theory, but needs certain fundamental mathematical knowledge as a vehicle.

The writer does not contend that the present high school mathematics program is the best arrangement for preparation for the calculus. Other arrangements might yield far greater returns for the time and effort spent; but he has taken the situation as it is and has attempted to show what parts of secondary mathematics and analytic geometry are used in the solution of the problems of the calculus.

THE SCOPE OF THE STUDY

This study is limited in scope to mathematical facts, principles, formulas, and operations used in the solution of the problems of the calculus. No attempt is made to give a rigid definition of these terms, but a brief explanation may bring out the meaning that each term is intended to convey.

The terms *facts* and *principles* include axioms, algebraic rules or laws, geometric propositions, and equations of standard curves.

By *formula* is meant any mathematical relationship, scientific law, or rule that is expressed in mathematical shorthand.

By *operation* is meant any mathematical process that has been developed in the field of secondary mathematics and analytic geometry. No listing is made of operations that are purely arithmetical, such as addition, subtraction, multiplication, and division of whole numbers or fractions, or both, unless some algebraic notion, such as signed numbers, is introduced. In other words, no count or listing is made of the skills of arithmetic. They are assumed as necessary foundations for the algebraic techniques. Although these skills are very important, no enumeration of them is made, since the writer is primarily concerned with that part of secondary mathematics necessary to understand the calculus.

TEXTBOOK USED

In order to determine to what extent secondary mathematics was used in the solution of the problems of the calculus, it was necessary to solve and analyze the problems of a standard textbook of the calculus. *Elements of the Differential and Integral Calculus* by

Granville, Smith, and Longley is the textbook used as the basis for this study. This book was selected because it has been, since its publication in 1904, and still is, extensively used as a college textbook. It covers the topics completely and contains a wide range of problems. A revised edition was published in 1929. In the preface is the following statement by Professors Smith and Longley, who revised the original Granville textbook. "The labor of the authors will be amply repaid if this revised edition meets with the generous and well-nigh universal favor accorded Granville's *Calculus* since its first appearance." The hope of the authors has been realized; this revised edition was used as a textbook in 366 colleges and universities in 1932–33. The calculus textbook which was next highest in use in the preceding year was used in 185 colleges and universities.[1]

The textbook contains 513 pages and comprises 27 chapters. Four of the chapters are not included in this study for the following reasons: Chapter XVII, Reduction Formulas and Use of Tables of Integrals, was omitted because the problems involve substitution in certain formulas of the calculus to reduce them to expressions that can be integrated by ordinary methods, or by use of the table of integrals; Chapter XXI, Ordinary Differential Equations, was omitted because this topic is taught as a separate course in mathematics as work beyond the calculus; and Chapter XXII, Partial Differentiation, and Chapter XXIII, Applications of Partial Derivatives, were omitted on the assumption that the methods used in these chapters would be similar to those used in ordinary differentiation.

PROCEDURE

The lists of facts, principles, formulas, and processes involved in this study were compiled from a detailed analysis of the solution of the 2,811 problems from Granville's *Calculus*. The recorded frequency of the facts and principles represents the number of times that each was used, whether in the same problem or in different problems.

The formulas listed are those in the field of secondary mathematics that were used in the solution of the calculus problems. No listing was made of the formulas used in differentiation and integration, as they are beyond the field of secondary mathematics

[1] These figures were obtained from the publishers of these textbooks.

and belong to the calculus alone. They are listed in the textbook on which this study was based.

Listing the mathematical processes used was by far the greatest task. Each problem had to be solved in detail and analyzed. Each operation that was not purely arithmetical was recorded. A count was made of the number of times each was used; if the same operation appeared more than once in a problem, it was listed as often as it appeared.

In problems that had subheads with parts designated as (a), (b), (c), etc., all parts were solved individually and each part was treated as a separate exercise. There were many problems of this type. The greatest number of subheads of any one problem was 18. There were 1,994 problems, not counting the parts separately. The total number when all parts were included was 2,811. The problems that had more than one part, but did not have the parts designated as (a), (b), (c), etc., were counted as only one exercise.

In the case of many problems, more than one solution was possible; only one, however, was made. If a different choice of solution had been made, the frequency table in all probability would have been somewhat different from the one given in this study. In any group of problems that followed a given topic, the solution used, where a choice was possible, was the one that seemed to follow naturally in order to illustrate the topic. For instance, under the heading of fractions $(x+3)/(x-2)^2$ was a problem differentiated as a quotient, and not as a product, as was possible by writing it in this form $(x+3)\ (x-2)^{-2}$. This procedure was consistently followed throughout both by the writer and by those who assisted in the work.

In any study that is made independently by one person certain methods will be used over and over again. This narrows the subject in scope, for the processes used will be limited to the ones that the investigator is accustomed to use. To avoid this repetition of method the writer asked four capable students of mathematics to assist in part of the work. Assistant A, a senior in college and an honors student in mathematics, solved the problems in three chapters. Assistants B and C, students doing graduate work in mathematics, who had studied mathematics for four years in college, worked together and solved the problems in six chapters. Assistant D, also a graduate student in mathematics and a candidate for the degree of Doctor of Philosophy, solved the problems in

two chapters. The writer checked the work of the assistants, solved the problems in the remaining chapters, and did the analysis for all the problems. It was the aim of all who helped with the problems to solve each one by the method that would best illustrate the topic under which it was listed. In this way the general method of attack was the same but the details of solution varied with the individual, thus giving a more representative list of processes than would have been obtained by one person.

The results of the analysis are classified under four different headings: algebra, geometry, trigonometry, and analytic geometry. Each division is discussed in a separate chapter, together with the tabulated results. A separate table presents the data on each major topic. By this means it is hoped that the results will be more clearly presented, and the importance of any subtopic can be judged by its frequency when compared to kindred topics. For example, under the table, Fundamental Operations, the relative importance, based on frequency, of multiplying a binomial by a monomial as compared to the product of two binomials can be seen at a glance. Moreover, since more than one hundred algebraic facts and operations are listed, the table would be too long to be easily and conveniently read if these were not grouped by topics.

There may be some overlapping in the divisions of the chapters, as well as in the grouping into tables, but in a general way each fact and operation is listed as algebraic, geometric, trigonometric, or belonging to the field of analytic geometry.

The purpose of this investigation is not to classify facts and operations into water-tight compartments, but rather to show how much and to what extent secondary mathematics is used in solving the problems of the calculus, and to compare the findings with the general requirements in the various courses of secondary mathematics.

CHAPTER II

ALGEBRA

In the solution of the 2,811 calculus problems the facts and operations from the field of secondary mathematics and analytic geometry were distributed in the manner indicated in the following table:

TABLE I

FACTS AND OPERATIONS USED IN THE SOLUTION OF THE CALCULUS PROBLEMS

Subject	Number of Facts and Operations	Frequency	Percentage
Algebra	129	26,576	88.6
Geometry	43	290	1.0
Trigonometry	37	2,237	7.5
Analytic geometry	67	868	2.9
Total	276	29,971	100.0

The 26,576 facts and operations from the field of algebra have been classified into ten groups and each presented in a separate frequency table. The main groups or divisions appear in the following order:

		Frequency
Table II	Fundamental operations	5,556
Table III	Laws and axioms	6,499
Table IV	Fractions	1,766
Table V	Factoring	1,003
Table VI	Radicals	940
Table VII	Logarithms	329
Table VIII	Graphs	219
Table IX	Equations	646
Table X	Substitutions	6,204
Table XI	Miscellaneous abilities	3,414

The following specimen sheet of the analysis is presented as an illustration of the method that was employed in determining what facts and operations had been used in the solution of the problems.

SPECIMEN SHEET OF ANALYSIS

Integrate the following: $y=\int\frac{dx}{\sqrt{ax}}$

(1) Break up an expression under a radical into separate radical factors.	(1) $y=\int\frac{dx}{\sqrt{a}\sqrt{x}}$
(2) Distinguish between constant and variable.	(2) $y=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x}}$
(3) Change a root to an expression with fractional exponents.	(3) $y=\frac{1}{\sqrt{a}}\int\frac{dx}{x^{\frac{1}{2}}}$
(4) Change a fraction to an expression with a negative exponent.	(4) $y=\frac{1}{\sqrt{a}}\int x^{-\frac{1}{2}}dx$
(5) Choose correct formula and substitute in it.	(5) $y=\frac{1}{\sqrt{a}}\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$
(6) Combine signed numbers.	(6) $y=\frac{1}{\sqrt{a}}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$
(7) Divide a variable by a fraction.	(7) $y=\frac{2x^{\frac{1}{2}}}{\sqrt{a}}+c$
(8) Bring an expression with fractional exponent under a radical.	(8) $y=\frac{2\sqrt{x}}{\sqrt{a}}+c$
(9) Multiply both numerator and denominator by the same quantity.	(9) $y=\frac{2\sqrt{x}\sqrt{a}}{\sqrt{a}\cdot\sqrt{a}}+c$
(10) Find the product of a radical expression.	(10) $y=\frac{2\sqrt{ax}}{a}+c$

This is a very simple problem in integration, and the result might have been obtained mentally; yet, in order to list every operation, every step was worked out and the corresponding operations were noted. The same type of analysis was used throughout. This particular problem required a total of ten steps in the analysis. Some of the longer and more difficult problems required a great many more.

After solving some of the problems it was necessary to perform additional operations in order to get the result in the form in which the textbook answer was given.

Only one classification was given to any one operation, even if it could have been broken up into several parts. For example, $(x-3)$ $(x+4)$ was simply called the product of two binomials. It was taken for granted that the ability to multiply two binomials having a common term carried with it knowledge of the law of

exponents and the law of signs in multiplication. Mention of the law of signs was made only when that was the only operation that applied. In such an expression as $3\ (5-x)^2\ (-1)$ the answer obtained was $-3\ (5-x)^2$; the law of signs therefore applied here. The law of exponents was counted only when it was the primary fact involved, as in the exercise $x\ (x^2+3)\ (2x)$, which gave $2x^2\ (x^2+3)$ as a result. If the answer was to be further simplified by multiplying the factors together, then another step was necessary and the product of a monomial and a binomial would have been the operation that would apply to this extra step. This scheme was rigidly followed; hence the totals of the law of signs, the law of exponents, and other operations may appear small in comparison with the number of times they actually have been used.

The fundamental operations, as shown by Table II, on page 10, had a frequency of 5,556. These included all instances of addition, subtraction, multiplication, and division of integral quantities.

The use of addition was limited to finding the sum of signed numbers and to the combination of similar terms. The few cases of removing parentheses, preceded by a positive sign, where the parentheses actually existed, was almost an artificial set-up, as shown by the following example:

$$\text{If}\quad y=(3+x^2)^2,\quad \text{then}\quad y+\Delta y=[3+(x+\Delta x)^2]^2.$$

In problems where parentheses were understood to exist, as in the sum of positive fractions, where the bar separating the terms of the fraction served as parentheses for both numerator and denominator, no count was made of the cases, for this operation in no way affected the procedure.

The type of subtraction that appeared most often was algebraic subtraction, which in reality might have been classed as addition of signed numbers, except for the fact that in differentiating a power 1 is always subtracted from the exponent, for if $y=(x+3)^{\frac{1}{3}}$, then the derivative of y in respect to x is $\frac{1}{3}(x+3)^{\frac{1}{3}-1}$ which is $\frac{1}{3}(x+3)^{-\frac{2}{3}}$. Removing symbols of aggregation whether expressed or implied, when preceded by a negative sign, ranked high in this group. The most common types were not enclosed in actual parentheses but were the numerators of negative fractions where it was necessary to find the lowest common denominator for either clearing fractions or combining fractions under a common

TABLE II

FUNDAMENTAL OPERATIONS

	Frequency
Addition	
Adding signed numbers and combining similar terms	1,284
Removing parentheses preceded by a positive sign	4
Removing parentheses, preceded by a positive sign, when enclosed in other parentheses	5
Subtraction	
Removing parentheses preceded by a negative sign	235
Removing parentheses, preceded by a negative sign, when enclosed in other parentheses	2
Inserting parentheses following a negative sign	12
Algebraic subtraction	308
Multiplication	
Multiplying two monomials	965
Multiplying a monomial and a polynomial	1,154
Multiplying two binomials or a binomial and another polynomial	238
Multiplying two polynomials (neither a binomial)	16
Multiplying any quantity by zero	238
Expansion by the binomial theorem	385
Cube of trinomial	2
Division	
Dividing one monomial by another	323
Dividing a polynomial by a monomial	182
Dividing a polynomial by a trinomial	19
Dividing one polynomial by another (neither a binomial)	1
Dividing a binomial by a polynomial	7
Dividing a monomial by a binomial	19
Dividing a variable by a numerical fraction	157

denominator. Inserting a quantity in parentheses, following a negative sign, was used only a few times.

Multiplying by a monomial led the list in multiplication. The most common use of multiplying a polynomial by a monomial was in differentiating as

$$y=(x^2+3)(5-3x^3);\text{ then }\frac{dy}{dx}=(x^2+3)(-9x^2)+(5-3x^3)(2x).$$

And also in integrations of the following type:

$$\text{if } y=\int(3x^2-4x-3)dx;\text{ then } y=\int[3x^2dx-4xdx-3dx],$$

where the dx is treated as a monomial. Multiplication of a polynomial by a binomial was used throughout the textbook.

The multiplication of two polynomials where neither was a binomial was rare. Multiplying by zero appeared most often in clearing equations of fractions, where one member of the equation was composed of a fraction and the other was zero. Expansion by the binomial theorem included only squares and cubes of a binomial, but the square of a binomial predominated. The cube of a trinomial appeared but twice.

The division of two monomials was classed as the law of exponents in division. The most common usage of dividing a polynomial by a monomial, or a polynomial by a binomial, or a monomial by a binomial, or a binomial by a polynomial was in splitting up a fraction for differentiation or integration as shown by these three examples.

$$\text{If}\quad y=\frac{x^2+2x-3}{x};\ \text{then}\quad y=x+2-\frac{3}{x}.$$

$$\text{If}\quad y=\frac{x}{x+2};\qquad \text{then}\quad y=1-\frac{2}{x+2}.$$

$$\text{If}\quad y=\frac{x^2+1}{x^2+2x+3};\ \text{then}\quad y=1-\frac{2x+2}{x^2+2x+3}.$$

Division of a polynomial by a binomial was limited to nineteen cases, and the quotient of two polynomials, each consisting of more than two terms, was used but once.

The process of the reduction of fractions to lowest terms where both numerator and denominator contained a common factor, was not classed under division, for the process did not involve actual division. This operation is included in Table IV, Fractions, on page 13.

Table III, Laws and Axioms, on page 12, shows the frequency of the common laws of algebra and the axioms used in this analysis. The frequency of the law of signs in multiplication was four times as great as that of the law of signs in division. Under the laws of exponents, those of multiplication led the list, followed by those of raising to powers, division, and extracting roots, in the order named. The frequencies in these two groups of laws appear small when considered in relation to the totals for the 2,811 problems. The only cases recorded under this heading are those that were primarily an application of the law of signs, or law of exponents, as set forth in the first part of this chapter.

TABLE III

LAWS AND AXIOMS

	Frequency
Law of Signs	
Multiplication	1,172
Division	279
Law of Exponents	
Multiplication	843
Division	323
Raising to powers	556
Extracting roots	125
Rule of Proportion: If four quantities are in proportion they are in proportion by	
Inversion	14
Alternation	4
Axioms	
If equals are added to equals, their sums are equal	317
If equals are subtracted from equals, their remainders are equal	567
If equals are multiplied by equals, their products are equal	210
If equals are divided by equals, their quotients are equal	719
Like powers of equals are equal	389
Like roots of equals are equal	252
In an equation or inequality equals may be substituted for equals; Things equal to the same thing are equal to each other	729

The only rules of proportion used were those of inversion and alternation.

The use of the axioms was in the solution of equations, whether it was in solving for one unknown in terms of the other, or for substitution in simultaneous equations, or for finding the roots of equations. The 317 occurrences of the axiom of adding equals to equals were cases of adding a positive number to both sides of an equation; the 567 occurrences of subtracting equals from equals were cases where a negative number was added to both members of the equation. The 729 occurrences of the last axiom listed in Table III represent only the cases where one variable was substituted for another variable, or where two variables were equal to a third variable. The many cases of numerical substitution are not included in this table but are listed in Table X, which is given on page 22.

Table IV, on page 13, shows to what extent fractions were employed in the analyses of problems in this study. In many in-

TABLE IV

FRACTIONS

	Frequency
Reductions of fractions to lowest terms by canceling	
Common monomial factor	627
Common binomial factor	149
Common trinomial factor	5
Division of both numerator and denominator by the same variable	37
Multiplication of both numerator and denominator by the same number	158
Square root of a fraction when it contains at least one variable	20
Addition of fractions	
Having the same denominators	18
Having unlike denominators where it was necessary to find lowest common denominator	247
Multiplication of fractions where no factoring was necessary	22
Division of fractions	
Where no factoring was necessary	307
Where factoring was necessary	2
Clearing of equations of fractions	74
Simplification of complex fractions	100

stances it was advantageous to reduce the fractions to a simpler form by dividing out the common factor. The common monomial factor was used more than four times as frequently as the other two forms. This operation was not peculiar to any one set of problems, but was found throughout the textbook among various problems. These operations might have been classed as dividing both numerator and denominator by the same quantity, but since no actual division took place they are classed only as reduction of fractions to lowest terms by canceling the common factor. These listed frequencies do not include reduction by dividing both terms of the fraction by a constant, inasmuch as this is arithmetical rather than algebraic.

Dividing both numerator and denominator by the same variable where there was no common factor appeared in only three groups of problems; the most common usage was as follows:

Prove the following statement

$$\lim_{x \to \infty} \frac{5-2x^2}{3x+5x^2} = -\frac{2}{3}.$$

In order to solve the problem it was necessary to divide both numerator and denominator by x^2, getting

$$\frac{\dfrac{5-2x^2}{x^2}}{\dfrac{3x+5x^2}{x^2}} \quad \text{which equals} \quad \frac{\dfrac{5}{x^2}-2}{\dfrac{3}{x}+5}.$$

The variable terms then came under the classification of certain special limits listed in the textbook.

Multiplying both numerator and denominator by the same number might have been classed as arithmetic, as a constant was usually the common multiplier; but since it had specific application to problems of integration it is included. The following will illustrate the use:

$$\text{Given} \quad y=\tfrac{3}{4}\int(x^2-10)^3x\,dx.$$

Before this problem could be integrated it had to be made into an integrable expression by introducing 2 as a multiplier; that is, multiplying each term of a fraction by the same number. The expression then became

$$y=\tfrac{3}{4}\cdot\tfrac{1}{2}\int(x^2-10)^3 2x\,dx,$$

and the integration could be performed.

Splitting up a fraction could have been repeated in this table, since it was of fractional form before division, but inasmuch as it comes primarily under the heading of division it is listed in Table II.

In only twenty instances was it necessary to take the square root of a fraction that contained at least one variable. These came in the solution of such equations as

$$y=\sqrt{\frac{4}{x^2}}; \quad \text{therefore} \quad y=\pm\frac{2}{x}.$$

The phrase "addition of fractions" includes the combining of fractions that were separated by either a positive or a negative sign. Where the denominators were the same the cases were limited to less than twenty. The more general usage occurred among problems necessitating the finding of the lowest common denominator in order to combine the fractions; none of these contained more than three terms. Addition of fractions is scattered throughout the textbook and is not limited to any specific group of problems.

Under multiplication and division of fractions there were only two examples where it was possible to factor before performing the indicated operation. In most cases of multiplication, or division changed to multiplication by inverting the divisor, it was a matter of simply multiplying the numerators in order to get the numerator of the product, and similarly for the denominators.

There were seventy-four equations whose solutions involved clearing of fractions; in the main, they were composed of terms, of which one was a fraction and the other a whole number, or both members were fractions. Among these fractional equations were none of more than three terms, and most of them consisted of two fractional terms and one integral term.

The 100 cases of complex fractions occurred in simplification of problems in differentiation and integration or in the solving of equations. More than two-thirds of them appeared in problems of differentiation. They ranged from very simple to the more complicated forms and were roughly grouped into the three following types:

$$\text{(a)}\ \frac{\dfrac{1}{2x^{\frac{1}{2}}}}{\dfrac{1}{2y^{\frac{1}{2}}}},\quad \text{(b)}\ \frac{1+\dfrac{1}{(x+1)^{\frac{1}{2}}}}{4-\dfrac{1}{(x+1)^{\frac{1}{2}}}},\quad \text{(c)}\ \frac{\dfrac{x}{2(x+1)}+(x+1)^{\frac{1}{2}}}{(x+1)}.$$

Their simplification involved division of fractions, multiplication of fractions, and combining fractions by finding the lowest common denominator.

TABLE V

FACTORING

Factoring a quantity by removing	Frequency
The common monomial factor	793
The common binomial factor	8
The common polynomial factor	1
Factoring the difference of two squares	46
Factoring quadratic trinomials	129
Factoring an expression of four terms by grouping	6
Factoring the sum or difference of two cubes	13
Factoring by using the factor theorem	7

The frequency of the uses of the various types of factoring is shown in Table V. Removing the common monomial factor was

by far the most frequent, being used nearly four times as often as all other cases combined. Removing the common binomial factor occurred only eight times, and removing the common trinomial factor only once.

The cases of factoring a quadratic trinomial included both the quantities that were perfect squares and those that were not. The ratio of these two was about one to one, there being sixty cases of the former and sixty-nine of the latter.

Factoring the difference of two squares ranked next in frequency, followed by the cases of the sum or difference of two cubes. There were six cases of factoring by grouping and only seven where the factor theorem was used.

The use of factoring in solving problems of multiplication and division of fractions was almost negligible, as is shown in Table IV, on page 13. Attention is again called to this fact in order to show that this was not the chief application of factoring; its primary use was to reduce fractions to lowest terms by removing the common factor.

TABLE VI

RADICALS

	Frequency
Extracting the square root of a	
Monomial that is a perfect square	278
Polynomial that is a perfect square	60
Extracting the cube root of a monomial that is a perfect cube	21
Product of two radicals	128
Quotient of two radicals	38
Rationalizing fractions having	
Monomial denominators	27
Binomial denominators	1
Rationalizing the numerator of a fraction	1
Removing rational factor from under the radical—as $\sqrt{2x^2}=x\sqrt{2}$	266
Putting rational factor under the radical	4
Solving radical equation	9
Breaking up an expression under radical into separate irrational factors as $\sqrt{2x}=\sqrt{2}\sqrt{x}$	107

Table VI shows to what extent radicals were used. The most common usage was extracting the square root of a monomial, which was used in finding the roots of an equation of the following type:

$$y^2=4, \quad \text{where} \quad y=\pm 2.$$

Extracting the cube root of a monomial had similar usage. Of the 129 cases of factoring quadratic trinomials listed in Table V, on page 15, sixty were examples of extracting the square root of a trinomial that was a perfect square. Most of these were used in drill examples of integration.

Finding the product of two radicals was used in differentiating a product of this type:

$$\text{If } y=x(x^2+3)^{\frac{1}{2}}, \text{ then } \frac{dy}{dx}=\frac{x^2}{(x^2+3)^{\frac{1}{2}}}+(x^2+3)^{\frac{1}{2}},$$

which, when simplified by finding the least common denominator and adding, involved the product of two radicals.

The quotient of two radicals included the root of a quotient as well as the quotient of the roots, for example,

$$y=\sqrt{\frac{x+3}{x-5}}$$

may be written

$$y=\frac{\sqrt{x+3}}{\sqrt{x-5}},$$

which was used when the problem was differentiated logarithmically for

$$\log y = \tfrac{1}{2} \log [(x+3) - \log (x-5)].$$

It was necessary to rationalize only twenty-eight denominators; of these, twenty-seven were monomials. In one case only was the rationalization of the numerator needed.

The breaking up of an expression under a radical into factors, at least one of which was rational, occurred often. The following is a typical example:

$$\sqrt{9x}=3\sqrt{x}.$$

The reverse of this operation, converting an expression composed of a rational factor and an irrational factor into an expression under a radical, occurred but four times. For example:

$$a\sqrt{x}=\sqrt{a^2x}.$$

In a radical expression composed of a constant and a variable where neither was rational, it was necessary to separate the constant from the variable before solving. This operation occurred one hundred seven times.

Only nine equations containing radicals in both members of the equation appeared, but no equation contained more than two radical terms. Expressions of this type:

$$\text{If } y=\sqrt{y+2}, \text{ then } y^2=y+2,$$

were not classed as radical equations, but were listed under the axiom "equal powers of equals are equal," as given in Table III, on page 12.

The use of logarithms, as shown in Table VII, was confined mostly to drill examples in differentiation and integration. If the laws of exponents are thoroughly mastered, then the application of logarithms to the solution of problems involving multiplication, division, involution, and evolution can easily be made.

Ability to use both the table of Common or Briggsian logarithms and the table of Natural or Napierian logarithms was needed, and, along with this, ability to interpolate in each case. Only a few instances of conversion from natural or base *e* logarithms to common or base *10* logarithms appeared in the solution of the calculus problems.

TABLE VII

LOGARITHMS

	Frequency
The use of logarithms in	
Multiplication	30
Division	99
Raising to powers	67
Extracting roots	7
Use of tables of logarithms	
Base *10*	8
Base *e*	62
Interpolation in logarithms	16
Converting the logarithm of a number from base *e* to base *10*	3
Solving an equation logarithmically by taking log of both members	37

Problems in differentiation of this type:

$$y=\sqrt{\frac{x+1}{x-1}}$$

were solved logarithmically for

$$\log y=\tfrac{1}{2}[\log (x+1)-\log (x-1)].$$

They were, therefore, classed under the heading of taking the logarithm of both members of an equation.

Table VIII lists the algebraic curves that had to be plotted in detail in order to solve the equations. It was necessary to plot thirteen linear, twenty-five quadratic, nineteen cubic, and four biquadratic curves in order to solve a special group of thirty-nine equations graphically.

TABLE VIII

GRAPHS

	Frequency
Plotting linear equations	13
Plotting quadratic equations	25
Plotting cubic equations	19
Plotting biquadratic equations	4
Solving simultaneous equations graphically	39
Interpolating in graphic solution of simultaneous equations	39
Finding x-intercept and y-intercept	80

These thirty-nine equations were such that, by transposing some of the terms, both members of the equation could be made to represent the equation of a standard curve, and then each of these members was equated to a new variable. In this way each equation was changed to a pair of simultaneous equations. This method is explained fully in the textbook. The other plotted curves resulting from these problems were not included in this table, for they were not algebraic curves. They are shown in Table XIX, on page 41, which lists the curves in connection with the uses of analytic geometry presented in Chapter V.

In order to set up the limits in various problems of integration it was necessary to find the x-intercept, the y-intercept, or both. This operation shows a frequency of eighty.

Table IX shows the 336 equations listed under the heading, "Solution of equations having one unknown." These equations were used in the solution of problems which were distributed throughout the textbook. The linear equation had the greatest frequency, followed by the complete quadratic, and then the quadratic in two terms. The cubic equation in two terms occupied fourth place. The others listed were used only a few times.

The solution of simultaneous equations in one unknown was limited to eighty-five cases. These equations contained a variety

TABLE IX

EQUATIONS

	Frequency
Solution of equations having one unknown	
Linear equations	150
Quadratic equations consisting of two terms	43
Quadratic equations consisting of three terms	90
Cubic equations consisting of two terms	26
Cubic equations—all other forms	8
Biquadratic equations consisting of two terms	4
Biquadratic equations consisting of three terms of quadratic form	4
Biquadratic equations—all other forms	6
Equations higher than fourth degree	5
Solution of simultaneous equations having two unknowns	
Two linear equations	6
One linear and one quadratic equation	33
One linear and one radical equation	1
Two quadratic equations	33
One linear and one cubic equation	4
One quadratic and one cubic equation	7
Two cubic equations	2
Solution of simultaneous linear equations	
Equations having three unknowns	25
Equations having four unknowns	20
Equations having five unknowns	1
Equations having six unknowns	2
Equations having eight unknowns	1
Solution of equations that contain	
Exponential terms	16
Logarithmic terms	4
Trigonometric terms	21
Solution of equations to represent the value of one unknown in terms of the other	55
Solution of simultaneous polar equations	23
Solution of equations by graphs	39
Solution of equations by Newton's Method	17

of combinations beginning with two linear equations and continuing through all possible combinations of linear, quadratic, and cubic equations to those containing two cubic equations. The one occurrence of a pair of simultaneous equations consisting of a linear and a radical equation was noted in the following problem:

$$x+y=1,$$
$$\sqrt{x}+\sqrt{y}=1.$$

The frequency of simultaneous equations containing three or more unknowns was forty-nine. These were used in the solution of problems involving partial fractions. Of this group there were twenty-five equations in three unknowns, twenty equations in four unknowns, one equation in five unknowns, two equations in six unknowns, and one equation in eight unknowns.

There were forty-one equations that contained exponential, logarithmic, or trigonometric functions that had to be solved.

It was often necessary, in equations containing two variables, to solve for one variable in terms of the other. For example:

$$\text{If} \quad 2xdx-2ydy=0, \quad \text{then} \quad dy=\frac{xdx}{y}.$$

There were twenty-three problems in integration that necessitated the solution of simultaneous polar equations in order to determine the limits.

The use of graphs in the solution of equations was needed only when problems were so worded as to require a graphic solution. In equations of this type:

$$x^3-4x-8=0,$$

the terms $-4x$ and -8 were transposed and the equation written thus:

$$x^3=4x+8;$$

then it was written as two separate equations:

$$y=x^3,$$
$$y=4x+8.$$

and plotted on the same axes. By using this method of introducing the variable y all the thirty-nine equations were broken up so that both of the resulting equations were standard curves. These seventy-eight resulting equations represented curves that were linear, parabolic, cubical parabolic, biquadratic, trigonometric, exponential, and logarithmic.

Table X lists the various substitutions. The one most extensively used was substitution in a formula, after selecting the correct formula. No problem that is to be differentiated or integrated by rule can be solved until the correct formula is known. There were 2,734 substitutions in such formulas. This is the

TABLE X

SUBSTITUTION

	Frequency
Substituting a constant for a variable	2,681
Substituting one variable for another	522
Choosing correct formula and substituting in it	2,734
Substituting one variable in one equation in terms of another in the second equation	51
Substituting a new variable for given variable, and finally making reverse substitution	216

highest frequency for any single operation noted in this study. The following example will illustrate this use:

$$\text{Differentiate } y = e^{-x^2}.$$

The general type that suits this particular problem is:

$$y = e^u;$$

$$\text{therefore} \qquad \frac{dy}{dx} = e^u \frac{du}{dx}$$

is the general formula. Instead of the variable u the variable $(-x^2)$ was substituted in the general formula before the problem could be differentiated.

Substituting a constant for a variable showed a frequency of 2,681, which is next in rank to the frequency of substituting in a formula. In the final answers which were expressed in variables, it was frequently necessary to substitute a constant for the variable in order to find the value of the derivative at a given instant, or to find the numerical value of a problem in integration within certain limits; and it was also necessary when $f(x)$ was given and $f(c)$ required (c being some definite constant); hence this substitution had a very high frequency.

Substituting one variable for another was extensively used. Following are two illustrations:

(a) Given $f(x)$, find $f(a+1)$.

(b) $\dfrac{\sin x}{\cos x} = \tan x$.

It was quite often necessary to introduce a new variable for a given variable in order to solve the problem, and later, after solv-

ing the problem, make the reverse substitution. The following will serve as an illustration:

$$\int \frac{(\sqrt{x+1}+1)dx}{\sqrt{x+1}-1}.$$

Let $y=\sqrt{x+1}$, hence $y^2=x+1$, and $dx=2ydy$. Then the problem becomes

$$\int \frac{(y+1)2ydy}{y-1},$$

which when integrated gives

$$y^2+4[(y-1)+\log (y-1)].$$

By reversing the substitution we can get the final answer in terms of x.

$$(x+1)+4[\sqrt{x+1}-1+\log (\sqrt{x+1}-1)].$$

Substituting in simultaneous equations where it was necessary to eliminate one variable in order to solve ranked next. This substitution was made by solving for one variable in terms of the other in one of the equations and then making this substitution in the second equation. An example follows: If $v=r^2h$, and we wish to differentiate v in respect to r, then it is necessary to set up a relation between r and h, and substitute the value of h in terms of r in the original equation, giving v as a function of r only.

Miscellaneous abilities, shown in Table XI are almost self-explanatory, and only a few words of explanation and some definite illustrations will be given. Many problems contained constants that were literal; before the quantity was differentiated or integrated distinction had to be made between the constant and variable. The following are examples:

$$\text{Given} \quad y=2ax^2, \quad \text{find } \frac{dy}{dx};$$

$$\text{or} \quad \text{Given} \quad \frac{dy}{dx}=bx^4, \quad \text{find y.}$$

Finding the lowest common denominator was necessary in combining fractions by addition and subtraction, and also in clearing equations of fractions.

Problems containing fractional exponents or negative exponents

TABLE XI

MISCELLANEOUS ABILITIES

Ability to	Frequency
(a) distinguish between constants and variable when both are literal	1,079
(b) find lowest common denominator	321
(c) deal with fractional exponents	381
(d) change fractional exponent to radical form	375
(e) change radical form to fractional exponent	365
(f) deal with negative exponents	290
(g) convert a fraction to an integral expression containing negative exponents	161
(h) convert quadratic trinomial into special form of sum or difference of two squares by breaking up the absolute term	166
(i) convert expressions containing negative exponents to forms containing only positive exponents	118
(j) handle division in a fraction if the denominator approaches zero or infinity	50
(k) deal with these indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 1^{∞}, ∞^0, 0^0	108

together totaled 746. These could have been classed under laws of exponents, but ability to work with them is somewhat different from the ability required to work with positive integral exponents. The use of fractional and negative exponents was about equally distributed among problems of differentiation and integration.

Changing radical forms to quantities with fractional exponents was made necessary by most of the problems of this type being stated in radical form. These were converted to fractional exponential forms before solving, and were later changed back to the radical form, to get the answer as stated in the textbook. If the answers had been left in fractional exponential form, the frequency of converting to radical form would have been lessened.

Many fractions with variables in the denominator were made more usable by converting them into expressions containing no denominator. This was accomplished by moving the variables to the numerator and changing the signs of their exponents. Finally the reverse operation was performed to eliminate negative exponents.

Converting quadratic trinomials into special forms of the sum or difference of two squares, by breaking up the absolute term,

was a device that was used 166 times. The following example illustrates this procedure:

(a) $x^2-8x+25$ can be written as $(x^2-8x+16)+9$,

which is of the form $(x-4)^2+(3)^2$, the sum of two squares.

(b) $x^2+10x+16$ can be changed to $(x^2+10x+25)-9$,

which is the difference of two squares.

This scheme was used to convert expressions into the forms containing $\sqrt{a^2 \pm u^2}$ in order to integrate.

The ability to divide by a variable that approaches either zero or infinity as a limit was required fifty times. Application of this process appeared in problems in which it was necessary to find the limiting value of fractional expressions whose denominators approached zero or infinity.

The indeterminate forms showed a frequency of 108. They appeared in problems under the heading "Theorem of Mean Value and Its Application" in the textbook, and were confined exclusively to this one section. The fact that this topic is not usually treated in secondary mathematics textbooks may account for the fact that the methods used to evaluate problems containing these forms are fully given in the calculus textbook.

SUMMARY

In the summary of this chapter of the uses of algebra in the solution of the problems of the calculus no attempt is made to draw conclusions; for this summary is meant only as a résumé. The general summary, in Chapter VII, includes conclusions based on the entire study.

1. Among the fundamental operations the simplest cases of addition, subtraction, multiplication, and division of integral quantities showed a very much higher frequency than did the more complicated forms. The most complicated of the four fundamental operations listed in this chapter were finding the product of two polynomials, which was used sixteen times, and the division of a trinomial by a trinomial, which was used only once.

2. The law of signs and the law of exponents were extensively used; so also were the familiar axioms of algebra, dealing with addition, subtraction, multiplication, division, involution, evolution, and substitution of equals for equals. The use of the rules

of inversion and alternation in proportion was limited to fourteen and four times respectively.

3. Reducing fractions to lowest terms ranked highest in frequency among the operations dealing with fractions. In performing the fundamental operations with fractions, only simple cases occurred. No quantities were composed of more than three fractional terms. Among the complex fractions there was none whose numerator or denominator consisted of more than two terms.

4. Removing the common monomial factor was the most frequently used operation in factoring, followed by factoring a quadratic trinomial and factoring the difference of two squares. Other methods of factoring were seldom used.

5. The radical was most frequently used in extracting the square root of a monomial that was a perfect square and removing the rational monomial factor from under the radical. The product of two radicals ranked third, followed by factoring an irrational expression into two irrational factors. The use of rationalizing the denominator and solving radical equations was infrequent.

6. The chief application of logarithms appeared in multiplication, division, and raising to powers; using the table of natural logarithms; and solving equations logarithmically.

7. The only algebraic curves that had to be plotted in detail were linear, quadratic, cubic, and biquadratic equations; they appeared in the graphic solution of simultaneous equations.

8. In equations of one unknown, linear and quadratic equations were the only ones extensively used. In simultaneous equations in two unknowns, those composed of one linear and one quadratic equation, or those composed of two quadratic equations, were the ones most commonly used. Among the simultaneous equations in three or more unknowns only four were composed of more than four unknowns.

9. Choosing the correct formula and substitution in it, and substituting a constant for a variable, ranked high in frequency. Substituting one variable for another was a frequent device.

10. Among the miscellaneous abilities those dealing with fractional and negative exponents, radical forms, distinguishing a constant from a variable when both were literal, and finding the least common denominator, showed a high frequency in the solu-

tion of the problems. Next in frequency were the ability to convert a quadratic trinomial into a special form of the sum or difference of two squares and the ability to find the limiting value of a fraction as the denominator approaches zero or infinity.

In the uses of algebra as a whole in the analysis of the 2,811 problems of the calculus the operations were confined almost entirely to the simpler types. The more difficult forms appeared infrequently and the extremely complicated forms were rare.

CHAPTER III

GEOMETRY

As shown by Table I, on page 7, the frequency for the geometric facts was 290, which was 1.0% of the total number of facts and operations used in the analysis of the 2,811 problems. This seems a rather small percentage, but inasmuch as the problems whose solution depended on some geometric fact contained only one or two such facts, and at the same time contained many algebraic facts and operations, the ratio is not so low as it at first appears.

The uses of geometry were found chiefly among the problems whose solution depended on the various applications of the derivative, and particularly in problems dealing with maxima and minima.

The facts from geometry are presented in three tables, namely:

		Frequency
Table XII	Facts from plane geometry concerning triangles ..	92
Table XIII	Other facts from plane geometry	9
Table XIV	Formulas from (a) plane and (b) solid geometry ...	189

The total frequency of these facts was 290; of these more than two-thirds were from plane geometry. Among the formulas those from the field of solid geometry were used nearly as often as those from plane geometry, the ratio being 90 to 105.

There were no problems in which the compasses and straight edge were required for construction work. The nearest approach to this was drawing the circle of curvature to certain curves, and drawing tangents to curves at specified points.

Table XII lists the facts of plane geometry concerning triangles. The first group in this table consists of the theorems dealing with triangles in general. In those problems whose solution involved similar triangles this fact was used in order to establish a relationship between two varying quantities. In order to solve the following problem:

> Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone,

TABLE XII

GEOMETRIC FACTS CONCERNING TRIANGLES

	Frequency
Triangles in General	
Corresponding sides of similar triangles are in proportion	13
A line through two sides of a triangle parallel to the third side forms a triangle that is similar to the given triangle	12
The areas of two similar triangles are to each other as the squares on any two corresponding sides	1
Right Triangles	
The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides	57
The perpendicular from the vertex of the right angle to the hypotenuse of the right triangle is the mean proportional between the segments of the hypotenuse	4
If an acute angle of one right triangle is equal to an acute angle of another, the triangles are similar	1
In a right triangle the mid-point of the hypotenuse is equidistant from the three vertices	1
Isosceles Triangles	
In an isosceles triangle the angles opposite the equal sides are equal	1
In an isosceles triangle each acute angle is 45°	1
In an isosceles triangle the perpendicular from the vertical angle to the base bisects the base	1

it was necessary to set up a relationship between the radius of the inscribed cylinder and the radius of the given cone. These relationships were expressed as proportions. The theorem concerning the areas of two similar triangles was used but once. Among the sixty-three applications of geometric facts based on right triangles the Pythagorean theorem was used fifty-seven times. Its use was to set up relationship between variables in problems that involved areas, volumes, or implicit differentiation. Two of the other three geometric facts based on right triangles dealt with similar triangles, or proportions resulting from them. Only once was it necessary to know that the mid-point of the hypotenuse of a right triangle is equidistant from the three vertices. The application of facts concerning isosceles triangles was limited to three uses, each of which occurred once.

Additional geometric facts are listed in Table XIII. The three facts relating to a rhombus were taken from the same problem. Without this geometric knowledge the use of the derivative in this problem would have been impossible. Only three times was it necessary to know that a tangent to a circle makes right angles

TABLE XIII

GEOMETRIC FACTS

(in general)

	Frequency
Rhombus	
(1) The diagonals of a rhombus bisect each other	1
(2) The diagonals of a rhombus are perpendicular to each other	1
(3) The diagonals of a rhombus divide the rhombus into four congruent triangles	1
Circles	
(1) The perpendicular from any point on a circle to a diameter is the mean proportional between the segments of the diameter	3
(2) If a line is tangent to a circle it is perpendicular to the radius drawn to the point of contact	1
(3) Two circles which are both tangent to the same line at the same point are called tangent circles	1
General	
The lateral surface of a cone and the area of a circle, whose radius equals the slant height of the cone, have the same ratio as angle of the sector formed by the surface of the cone being flattened out and two right angles. (Surface: Area = angle of sector: 360°)	1

with the radius at the point of tangency. The definition of tangent circles was needed but once. Setting up a proportion between the ratio of the lateral surface of a cone to the area of a circle whose radius equals the slant height of the cone and the ratio of the angle of the sector formed by the lateral surface of the cone being laid out flat and 360 degrees was used only one

TABLE XIV (a)

FORMULAS FROM PLANE GEOMETRY

	Frequency
Areas	
The area of a circle ($A=\pi r^2$)	31
The area of a triangle ($A=\frac{1}{2}bh$)	28
The area of a rectangle ($A=bh$)	16
The area of an equilateral triangle ($A=\frac{s^2}{4}\sqrt{3}$)	4
The area of a trapezoid [$A=\frac{1}{2}a\,(b+b')$]	4
The area of a square ($A=s^2$)	4
Lengths	
The altitude of an equilateral triangle ($h=\frac{s}{2}\sqrt{3}$)	4
The perimeter of a rectangle ($P=2b+2h$)	3
The length of the slant height of a cone ($L=\sqrt{h^2+r^2}$)	3
The circumference of a circle ($C=2\pi r$)	1
The perimeter of a square ($P=4s$)	1

time. This particular proportion dealt with both an area of a plane figure and a surface of a solid; it could have been classed either as from the field of plane or solid geometry.

The formulas from plane geometry are listed in Table XIV(a), and those from solid geometry in Table XIV(b). Among the formulas from plane geometry those concerning areas showed a much higher frequency than did those concerning distances, and the formulas for volumes greatly exceeded those for surfaces.

TABLE XIV (b)

FORMULAS FROM SOLID GEOMETRY

	Frequency
Volumes	
Volume of a cylinder ($V=\pi r^2 h$)	22
Volume of a sphere ($V=\frac{4}{3}\pi r^3$)	14
Volume of a cone ($V=\frac{1}{3}\pi r^2 h$)	12
Volume of a rectangular solid ($V=lwh$)	6
Volume of a cube ($V=s^3$)	3
Volume of a frustrum of a cone $V=\frac{1}{3}\pi h\ (R^2+r^2+Rr)$	3
Volume of a prism ($V=bh$)	2
Volume of zone of one base $V=\frac{1}{3}\pi h^2\ (3r-h)$	2
Surfaces	
Surface of a cone ($S=\pi rh$)	9
Surface of a cylinder ($S=2\pi rh$)	5
Surface of a zone of one base ($S=2\pi rh$)	3
Surface of a hexagonal prism ($S=ph$)	1
Total surface of	
Rectangular solid $T=2\ (lw+lh+hw)$	3
Sphere ($S=4\pi r^2$)	3
Cube ($S=6s^2$)	2

These formulas were used mostly in problems where it was necessary to establish relationships between areas and perimeters or between surfaces and volumes. The greatest usage of these relationships was in problems requiring the finding of maximum volume, minimum surface, or minimum cost.

SUMMARY

The following paragraphs are merely summarizing statements of the uses of plane and solid geometry in the solution of the calculus problems. Conclusions for this chapter are presented in Chapter VII.

The uses of geometry in this investigation were confined chiefly to those propositions relating to triangles and to the propositions

that may be expressed as formulas for computing distances, areas, surfaces, and volumes of the most commonly used geometric figures.

1. The uses of the theorems of geometry, except those that could be represented as formulas in solving the problems of the calculus, were limited almost exclusively to (*a*) the relationship of the sides of similar triangles; (*b*) the relation of the hypotenuse to the other sides of a right triangle; and (*c*) certain properties of other triangles.

2. The applications of theorems dealing with quadrilaterals and circles were infrequent.

3. The formulas were from both plane and solid geometry and included practically all those dealing with perimeters and areas of plane figures and surfaces and volumes of ordinary solids.

CHAPTER IV

TRIGONOMETRY

THE frequency of trigonometric facts, abilities, and formulas used in this study was 2,137, which is 7.5% of the total frequency of the facts and operations from the field of secondary mathematics and analytic geometry. The relation of the uses of trigonometry to the other subjects is shown in Table I on page 7.

The facts and abilities from trigonometry are shown, with their frequencies, in two tables:

		Frequency
Table XV	Trigonometric formulas	932
Table XVI	Trigonometric abilities	1,305

The chief use of trigonometry in solving the problems of the calculus was in substituting one trigonometric function for another in order to simplify an expression, or in order to change a problem into a form by which it could be solved.

Among the formulas which were used, as listed in Table XV, the fundamental identities showed the greatest frequency. These are ranked in descending order to show the distribution and relative importance of each. Their total frequency was 634. The frequency of half-angle and multiple-angle formulas was 278. These formulas, like the fundamental identities, were used to make substitutions in order to convert an expression into a workable form.

TABLE XV

FORMULAS

	Frequency
Fundamental Identities	
$\text{Sin}^2 A + \cos^2 A = 1$	145
$\text{Sec}^2 A - \tan^2 A = 1$	105
$\text{Sec } A = \dfrac{1}{\cos A}$	77
$\text{Tan } A = \dfrac{\sin A}{\cos A}$	76
$\text{Csc } A = \dfrac{1}{\sin A}$	65

$\text{Cot } A = \dfrac{\cos A}{\sin A}$	57
$\text{Tan } A = \dfrac{1}{\cot A}$	40
$\text{Csc}^2 A - \cot^2 A = 1$	34
$\text{Cos } A = \dfrac{1}{\sec A}$	29
$\text{Cot } A = \dfrac{1}{\tan A}$	6
Half-angle and Multiple-angle Formulas	
$\text{Cos}^2 A = \frac{1}{2} + \frac{1}{2} \cos 2A$	81
$\text{Cos}^2 \dfrac{A}{2} = \frac{1}{2} (1 + \cos A)$	61
$\text{Sin}^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$	52
$\text{Sin } 2A = 2 \sin A \cos A$	48
$\text{Cos } 2A = \cos^2 A - \sin^2 A$	14
$\text{Sin}^2 \dfrac{A}{2} = \frac{1}{2} (1 - \cos A)$	13
$\text{Cos } 2A = 1 - 2 \sin^2 A$	5
$\text{Cos } 2A = 2 \cos^2 A - 1$	3
$\text{Tan } 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$	1
Sum and Difference Formulas	
$\text{Cos } (A - B) = \cos A \cos B + \sin A \sin B$	5
$\text{Cos } (A + B) = \cos A \cos B - \sin A \sin B$	4
$\text{Tan } (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$	2
$\text{Sin } (A + B) = \sin A \cos B + \cos A \sin B$	1
$\text{Sin } (A - B) = \sin A \cos B - \cos A \sin B$	1
$\text{Sin } A - \sin B = 2 \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$	1
Polar Coordinates	
$x = \cos A,\ y = \sin A$	3
Hyperbolic Functions	
$\text{Sinh } x = \frac{1}{2} (e^x - e^{-x})$	1
$-\text{Sinh}^2 x + \cosh^2 x = 1$	1
$\text{Cosh } x = \frac{1}{2} (e^x + e^{-x})$	1

These substitutions were of value, especially in problems of integration. The use of the formulas for the sum and the difference of two angles was limited to five different formulas, whose total frequency was fourteen. The chief function of these formulas, like the ones listed above, was conversion. Changing from rectangular to polar coordinates was a device used only three times. The three hyperbolic functions listed here were used only one time each.

The trigonometric abilities are listed in Table XVI below. Chief among these abilities were those necessary for finding the numerical value of a trigonometric function and for performing the inverse operation. Four abilities were used in finding the

TABLE XVI

ABILITIES IN TRIGONOMETRY

	Frequency
Ability to find the numerical value of the trigonometric function of	
1. Angles expressed in degrees for	
(a) Certain acute angles, viz., 0°, 30°, 45°, 60°, 90°	462
(b) Certain obtuse angles, viz., 120°, 180°, 210°, 240°, 360°, 360n°	100
(c) Other angles in general	57
2. Angles expressed in radians	
(a) Certain angles, viz., $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, 2\pi$	360
Ability to convert trigonometric functions to degrees	
1. When expressed in fractional form for certain angles, viz., 0°, 30°, 45°, 60°, 90°	84
2. When expressed in decimal fractional form for angles in general	194
Ability to convert angular measure expressed	
1. In degrees to radians	23
2. In radians to degrees	25

trigonometric function of an angle. They were those dealing with certain acute angles whose trigonometric function could be found from geometric drawings, those dealing with certain obtuse or reflex angles whose value could also be found geometrically, those whose values were found by the use of tables, and those angles whose values were expressed in radian measure. The inverse operation was used to convert certain fractional functions into specific angles, viz., 0°, 30°, 45°, 60°, and 90°, and to convert other fractional functions into angles not included in the above group. It was also necessary to convert angular measure expressed in degrees to angular measure expressed in radians and vice versa.

SUMMARY

The following statements summarize the findings concerning the uses of trigonometry in the solution of the calculus problems. Conclusions for this chapter appear in Chapter VII.

1. Of the trigonometric facts and abilities used in the solution of the calculus problems, the formulas showed the greatest

frequency. Of these formulas the fundamental identities ranked highest, followed by the formulas for half-angles and multiple-angles, and then by the formulas for the sum and the difference of two angles. This last-named group was used much less often than the other two. The chief use of the formula was in making substitutions in order to convert the problem into a form that could be more easily solved.

2. The abilities in trigonometry most frequently used were those dealing with the finding of the trigonometric functions of certain specific angles and with the finding of the functions of angles in general, while the inverse operations for both of these groups were used only about one-third as often. The ability to convert angular measure expressed in degrees to angular measure expressed in radians, and the inverse of this operation concluded the list of abilities.

The uses of trigonometry were confined to the most familiar identities and formulas, with one exception—the use of hyperbolic functions. Formulas for hyperbolic functions were used in only three of the 2,811 problems solved in this study. The trigonometric operations, like those from algebra, are limited almost entirely to the simple processes and to the use of the more commonly known formulas.

CHAPTER V

ANALYTIC GEOMETRY

THE solution of some of the problems of the calculus depended on the application of certain facts from analytic geometry. For this reason this subject was included in this study, even though it is not classed as a secondary school subject. The extent of its use is shown in Table I, on page 7. The frequency of the uses of facts from analytic geometry, as shown in this table, is 868, which is 2.9% of the total frequency of all facts and operations used in the entire study.

The knowledge of certain formulas of analytic geometry was essential for solving some of the problems, as was the knowledge of certain facts concerning the curves whose equations were given in the statements of the problems. These facts and their frequencies are listed in the following tables:

		Frequency
Table XVII	Formulas and equations	301
Table XVIII	Curves expressed in rectangular coordinates	493
Table XIX	Curves expressed in polar coordinates	74

Table XVII, on page 38, includes only the formulas and equations which the student was supposed to know; that is, those formulas and equations that were not given in the statement of the problem in the textbook. The formulas and equations are listed separately in this table, and are arranged in each group in the order of their frequency.

In the grouping a distinction was made between formulas and equations, since some of these are usually classed as formulas and others as equations. The use of the formula for the distance between two points ranked highest, but was only one point higher than the use of the formulas for the length of the subtangent and the subnormal. These in turn were followed by the formulas for the length of the tangent and the normal.

The equations used most often were those for the tangent and the normal. The slope-point form of the equation of a straight line could have been included with the equation of the tangent,

TABLE XVII

FORMULAS AND EQUATIONS

	Frequency
Formulas	
Formulas for the distance between two points	29
Formula for the length of the subtangent	28
Formula for the length of the subnormal	28
Formula for the length of the tangent	7
Formula for the length of the normal	7
Standard Equations	
Equation of the tangent	71
Equation of the normal	66
Equation of a straight line (slope-point form)	25
Equation of a straight line (two-point form)	17
Equation of a parabola	6
Equation of an ellipse	5
Equation of a circle	3
Equation of a hyperbola	2
Equation of an equilateral hyperbola	2
Equation of a cubical parabola	2
Equation of a semicubical parabola	2
Equation of a witch	1

since it was the same in form. A distinction was made, however, for the reason that when the equation of the tangent was wanted, it was specifically stated so in the problem. These were therefore listed separately from the general use of the slope-point form. The frequency of the two-point form of the equation of a straight line was seventeen. It was necessary to know the equation of eight standard curves, for they were not supplied in the problems. This group included the standard equations of the parabola, ellipse, circle, hyperbolas, cubical and semicubical parabolas, and the witch. The equation of the parabola had the highest frequency, which was six; and the equation of the witch the lowest, which was one.

The equations which are listed in Table XVII are those used in the solution of the problems, although not expressed in their wording. The equations listed in Table XVIII are the ones that were expressed in the problems. It was not necessary to plot all curves in detail but it was essential to know their general shape, for many problems dealt with the areas under a curve, the volume generated by a curve, and the area or volume common to two intersecting curves. The curves most frequently used were the parabola, straight line, and circle. Ranking next to these in

TABLE XVIII

CURVES EXPRESSED IN RECTANGULAR COORDINATES

	Frequency
Equation of the parabola	141
Equation of the straight line	61
Equation of the circle	57
Equation of the cubical parabola	25
Equation of the hyperbola	23
Equation of the ellipse	22
Equation of curves in factor form	22
Equation of the semicubical parabola	20
Equation of the cardioid	13
Equation of the hypocycloid	12
Equation of cubic curves in general	9
Equation of the cycloid	6
Equation of curves in radical form	6
Equation of curves in fractional form	6
Equation of the catenary	3
Equation of the witch	3
Equation of the cissoid	2
Equation of the tractrix	2
Equation of the companion of the cardioid	1
Equation of the folium of Descartes	1
Equation of the sine curve	22
Equation of the cosine curve	9
Equation of the tangent curve	8
Equation of the cosecant curve	1
Equation of the exponential curves	12
Equation of the logarithmic curve	6

use were the cubical and semicubical parabolas, hyperbolas, ellipses, and curves expressed in factor form, of which the following is an example:

$$x=(2-y)\ (1+y)^2.$$

The only other curves that showed a frequency of more than ten were the cardioid and the hypocycloid. The others listed in the first division of Table XVIII were used less than ten times each. Among these only six curves were used whose equations were expressed in radical form in the problem. The following will illustrate this type:

$$y=\sqrt{4+x^2}.$$

There were also six curves the equations of which were expressed in the textbook in fractional form; that is, y varies as a quotient as in this example:

$$y=\frac{x}{6-x}.$$

Included in this group were some discontinuous curves.

All the standard curves listed in this table were illustrated in the textbook in Chapter XXVI.

The second division of the table is devoted to the trigonometric, logarithmic, and exponential curves. The only curves of the trigonometric functions used were the sine, cosine, tangent, and cosecant of half-angles, whole-angles, and multiple-angles. The exponential curves had a frequency of twelve. Of these, eleven were of the base e raised to some variable power, and one was $y=10^x$. The logarithmic curves were used six times. Among the curves in the second division there were various combinations, such as

$$y=\sin x+\cos x.$$
$$y=\cos x-x^2.$$
$$y=e^x \cos 3x.$$
$$y=\log \sec x.$$
$$y=\frac{x^2}{4}-\tfrac{1}{2}\log x.$$
$$y=\frac{e^x+1}{e^x-1}.$$

It was necessary to have the ability to work with these combinations as well as with curves expressed by a single function in a single term.

Table XIX lists the curves which are expressed in polar coordinates. These, like the curves listed in Table XVIII, on page 39, were given in the statement of the problem, but it was necessary to know the general shape of the curves. They are ranked in the table in order of frequency. None had a frequency of ten, and only three were used more than five times. They were the circle, the lemniscate, and the four-leaved rose. The others had a frequency of four or less. All the curves expressed in polar coordinates were illustrated in the textbook except the thirty-one

TABLE XIX

CURVES EXPRESSED IN POLAR COORDINATES

	Frequency
Equation of the circle	9
Equation of the lemniscate	7
Equation of the four-leaved rose	6
Equation of the parabola	4
Equation of the logarithmic spiral	4
Equation of the spiral of Archimedes	3
Equation of the limaçon	3
Equation of the hyperbolic spiral	2
Equation of the ellipse	1
Equation of the eight-leaved rose	1
Equation of the three-leaved rose	1
Equation of the two-leaved rose lemniscate	1
Equation of the curve with cusp of second kind at origin	1
Equation of miscellaneous curves	31

miscellaneous curves. This group was composed mostly of curves expressed as a combination of a trigonometric function and a constant.

SUMMARY

The uses of plane analytic geometry in this study were limited to two groups of formulas or equations, that is, those that were not expressed in the wording of the problems in the textbook, but were necessary to be known before the problem could be solved; and those that were given in equation form in the statement of the problem. The equations that were implied were used only about one-third as often as those that were expressed. Among the curves whose equations were stated in the problems, those expressed in rectangular coordinates greatly exceeded in number those expressed in polar coordinates. Most of the curves used were those illustrated in the textbook under the heading "Curves for Reference." Only the most commonly known formulas and equations of plane analytic geometry were used in the solution of the problems of the calculus.

A limited number of facts from solid analytic geometry were used in the solution of the problems of the calculus, but the necessary information was supplied in the calculus textbook. The student could obtain these facts in the calculus class when they were needed; therefore, it was not necessary for him to know them before studying the calculus.

CHAPTER VI

SYLLABUS REQUIREMENTS

THE findings of the analyses were compared with the requirements in secondary mathematics as set forth in the syllabus of the College Entrance Examination Board, in order to determine how much of the required subject matter of secondary mathematics was actually used in the solution of the calculus problems.

The syllabus of the College Entrance Examination Board has been selected for the comparison in preference to other syllabi as it is a general requirement for the country at large, whereas the other available syllabi are used only in limited areas, such as particular state, county, or city school systems. Then, too, the College Entrance Examination Board exerts a powerful influence on the subject matter taught in the high schools, probably a greater influence than any other single factor. Textbook writers are aware of this and often point out that the contents of a particular textbook meets all the requirements of the College Entrance Examination Board. In many cases the subject matter presented in the textbook becomes the subject matter taught in the high school.

In the following summary the topics from each of the separate fields of secondary mathematics are arranged in the same order as they are in the syllabus of the College Entrance Examination Board. Excerpts concerning each of the topics are presented in order to show clearly the comparison of the uses of secondary mathematics in solving the calculus problems with the prescribed requirements in secondary mathematics.

ELEMENTARY ALGEBRA (PART I)

Algebra to Quadratics

1. FORMULAS

Requirements (as stated in the syllabus)

> The meaning, use, evaluation, and necessary transformations of simple formulas involving ideas with

which the pupil is familiar, and the derivation of such formulas from rules expressed in words.

Uses (as found in this study)

The understanding of simple formulas and the ability to use, transform, and evaluate them contributed to the solving of the calculus problems. The formulas used most extensively were the ones commonly developed in plane geometry for perimeters and areas and in solid geometry for surfaces and volumes. Formulas other than these, when needed in the solution of the problem, were supplied in the statement of the problems in the textbook.

Every problem of calculating the definite integral contained an application of evaluation in an equation. This process is the same as the process of evaluation in a formula.

If, in the work done with formulas in secondary mathematics, the general idea of the dependence of one variable upon another is repeatedly emphasized (as stressed in the "Notes to Teachers" in the College Entrance Examination Board Syllabus in Algebra), it will be invaluable to the student in studying the calculus because the functional relationship of one variable to another is a basic idea of the calculus. The uses made of the application of the formula in the solution of the calculus problems coincide completely with the requirements set forth in the syllabus.

2. GRAPHS

Requirements

The graph and graphical representation in general.
The construction and interpretation of graphs.

Use

The chief use of graphs in this study was not so much in the actual plotting of the curve as in the portrayal of its general shape so that it could be readily sketched. There were only a few problems which called for the graphic solution of simultaneous equations. These were of the special type where one member of the equation, as stated in the textbook, was zero; but before the curve was plotted some terms were transposed so that each member of the equation then represented the equation of a standard curve. Each of these members was equated to some variable and the two equations thus formed were plotted.

In order to set up limits in certain problems of integration it was necessary to find the x-intercept and the y-intercept.

The use made of graphs and graphic solution of equations in the solution of the calculus problems included all the requirements set forth in the syllabus. In addition there was the special method of solving an equation graphically by breaking it up into two equations. The process was explained in the calculus text-book and the method could be presented in the calculus class when needed.

3. NEGATIVE NUMBERS

Requirements

> Negative numbers, their meaning and use.
>
> This requirement includes the fundamental operations with negative numbers and the interpretation of a negative result in a problem.

Use

There was wide use of the above facts and operations in the solution of the calculus problems. There were applications of the fundamental operations dealing with negative numbers and many problems whose solution gave a negative result, the sign of which carries with it a specific meaning.

Under this heading comes the application of the laws of signs in multiplication and in division, and the change of signs of quantities which are preceded by a negative sign, either in the removing of symbols of aggregation or in the enclosing of a quantity in parentheses.

If that part of the syllabus "Notes to Teachers" concerning negative numbers is observed by high school algebra teachers "the relation of real numbers to points on a line should be made clear, the cardinal principle being that to every such number corresponds a point on the scale and conversely. In this way, the relation of positive to negative numbers, of integers to fractional and irrational numbers, and of approximations of the values of both rational and irrational numbers will be rendered intelligible." This relationship of numbers will be ample training to enable the student to interpret the meaning of results of problems, especially if the result is a negative, an irrational, or an approximate number.

4. LINEAR EQUATIONS

Requirements

> Linear equations in one unknown and simultaneous linear equations involving two unknown quantities, with verification of results. Problems.
>
> The coefficients of a single linear equation in one unknown quantity may be literal factors. In the case of simultaneous equations, literal coefficients are restricted to simple integral expressions, and to cases readily reducible to such expressions.

Use

The uses made of linear equations included all the above-named types and in addition the solution of simultaneous equations in three or more unknown quantities. This last-named group will be discussed under simultaneous equations of three unknown quantities in Part II, Section 4, of Elementary Algebra.

Among the linear equations containing fractions there were none whose common denominator could not be found by inspection. Included among the abilities relating to literal equations was that of solving for one unknown quantity in terms of another in both equations and formulas containing two unknown quantities.

5. RATIO

Requirements

> Ratio, as a case of simple fractions; proportion, as a case of an equation between two ratios; variations. Problems.

Use

The knowledge of the meaning and use of ratio and proportion was essential to the understanding of the calculus problems. It was necessary to know that if four quantities are in proportion, they are in proportion by alternation or by inversion. The knowledge of the principle of variation was basic to the understanding of the theory of the calculus.

Teaching ratio as a fraction, or as a quotient of two numbers, and proportion as a fractional equation, as the syllabus suggests, will give ample training for the use of ratio and proportion as needed in the solution of the calculus problems. Such terms as alternation, inversion, and the like, can be taught as the need arises.

6. ALGEBRAIC TECHNIQUE

Requirements

The essentials of algebraic technique including:

(a) The fundamental operations.
(b) Factoring.
(c) Fractions, including complex fractions of simple type.
(d) Numerical verification of results.

Use

(a) The "Notes to Teachers" in the syllabus states that pupils should not be called upon to perform long and elaborate multiplications or divisions of polynomials, but should have complete mastery of the types that are essential to subsequent work. If this suggestion is observed in the teaching of secondary mathematics, then the pupil will be sufficiently well prepared to perform the operations necessary to the solution of the calculus problems. This study revealed that only the simplest types of the fundamental operations with polynomials were used.

(b) The only types of factoring listed in the syllabus are:

(1) Monomial factors.
(2) The difference of two squares.
(3) Trinomials of the type x^2+px+q.

The types of factoring that were chiefly used were the ones listed above, though slight use was made of three other types:

(1) Factoring the sum or difference of two cubes.
(2) Factoring by grouping.
(3) Factoring by the use of the factor theorem.

Inasmuch as these types are rarely used, the method of factoring them can be taught when needed. Among the quadratic trinomials that had to be factored there were some of the type ax^2+bx+c, but they were greatly outnumbered by those of the form x^2+px+q.

(c) The fundamental operations with fractions were used throughout this study. The only cases of fractions occurring were simple ones; no fractional expressions were composed of more than two fractional terms. In the case of combining fractions by addition or subtraction, the lowest common denominator could be found by inspection. The cases of multiplication and division of fractions were of the type where factoring and subsequent

cancellation were not possible. The complex fractions arising in the solutions of the calculus problems were no more difficult than the type

$$\frac{\dfrac{x+2}{(x-1)^{\frac{1}{2}}}-(x-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}},$$

which is somewhat more difficult than the types listed in the syllabus.

(d) The practice of numerical verification in secondary mathematics will give a broader understanding of the meaning of (a), (b), and (c) above, and will be helpful in evaluating results of the calculus problems.

7. EXPONENTS AND RADICALS

Requirements

(a) The proof of the laws for positive integral exponents.

(b) The reduction of radicals, and evaluation of simple expressions involving the radical sign.

(c) The meaning and use of fractional exponents.

(d) A process of finding the square root of a number, but no process of finding the square root of a polynomial.

Use

(a) The laws of positive integral exponents were used often in multiplication, division, raising to powers, and extracting roots of quantities in the solution of the calculus problems.

(b) The reduction of radicals was confined to removing the rational factor from a monomial under the radical sign, and rationalizing the monomial denominator when the denominator was expressed as an indicated square root or an indicated cube root of an irrational number.

(c) Understanding the meaning of fractional exponents was important since fractional exponents appeared in the solution of problems throughout the text. The ability to change radical forms to expressions with fractional exponents and vice versa was essential.

(d) The need of finding the square root of numbers that were not obvious rational numbers was small, for many results of the calculus problems could be left in the radical form indicated.

The only cases where it was necessary to extract the square root of a polynomial were quadratic trinomials that were perfect squares; these were included under the topic factoring.

In addition to the above uses of radicals in this study, the knowledge that the root of a product is the same as the product of the roots and that the root of a quotient is equal to the quotient of the roots was essential in solving certain problems of the calculus.

The use of radicals in the solution of the calculus problems goes very little beyond the requirements set forth in the syllabus, and these additional facts or processes will be discussed under Exponents and Radicals in Part II of Elementary Algebra.

8. NUMERICAL TRIGONOMETRY

This topic is discussed under the heading Trigonometry.

ELEMENTARY ALGEBRA (PART II)

Quadratics and Beyond

1. QUADRATIC EQUATIONS

Requirements

Numerical and literal quadratic equations in one unknown. Problems.

Use

"The requirement includes the solution of the general quadratic equation

$$ax^2+bx+c=0,$$

the conditions for the reality and for the distinctness of the roots, and the formulas for the sum and the product of the roots. Simple cases in which x is replaced by z^2 or by a linear binomial, and problems leading to quadratics, are also included; furthermore: The interpretation of the graph of such an expression as x^2-3x+5, meaning thereby the graph of the corresponding *e* equation,

$$y=x^2-3x+5.\text{"}^{1}$$

Knowledge of all these facts and abilities will prove helpful to the student who is studying the calculus.

The "Notes to the Teachers" emphasizes the solution of the

[1] Syllabus of College Entrance Examination Board.

quadratic equation by completing the square. Equations of the form $ax^2+bx=0$ should be solved by factoring. Beyond this, factoring should be restricted to such simple cases as $x^2-5x+6=0$. The formula is not excluded as a method of solving quadratic equations; the writer used this method almost exclusively in solving complete quadratic equations.

The analyses revealed certain other equations of degree higher than the second. The cubics of the form $ax^3-bx^2=0$, or $ax^3-bx=0$, or $ax^3-b^3=0$ could all be solved by factoring. The biquadratics either consisted of two terms and could be solved by factoring, or consisted of three terms and were of the quadratic form. There were only five cases of the solution of equations higher than the fourth degree, and the methods needed to solve such equations can be presented in the calculus class when needed.

2. BINOMIAL THEOREM

Requirements

> The binomial theorem for positive integral exponents, with applications.

Use

The only uses of the binomial theorem were cases where $n=2$, or $n=3$. There were none higher. The requirements state that in general it will be sufficient to take n equal to 8, or less than 8.

3. SERIES

Requirements

> Arithmetic and geometric series.
>
> The requirements are limited to the formulas for the nth term, the sum of the 1st n terms, the value of such an infinite decreasing series as $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots.$, and to simple applications.

Use

The formulas for arithmetic and geometric progressions were not used in the solution of the calculus problems; one problem, however, did require the knowledge of the meaning of an arithmetic progression.

The chapter on series in the calculus textbook presents the fundamental facts about series, particularly the geometric and infinite series. The topic is treated much more fully than the

syllabus requires. Even if the student is not familiar with series, the calculus textbook explains the theory so fully that little or no knowledge of series is necessary before beginning this chapter. The inclusion of series in the syllabus serves as an introduction to the idea of the limit.

4. SIMULTANEOUS LINEAR EQUATIONS

Requirements

Simultaneous linear equations in three unknown quantities. The coefficients may be integers, numerical fractions, or algebraic monomials.

Use

The analyses in this study revealed not only simultaneous equations in three unknown quantities but almost as many in four unknowns. There were also one equation in five unknowns, two in six unknowns, and one in eight unknowns. The work of solving a system of these last-named linear equations was not difficult, for in no set of the equations comprising the system were all the unknowns present in each of the individual equations, so by making a few combinations or substitutions the value of the unknowns could be obtained.

5. SIMULTANEOUS EQUATIONS

Requirements

Simultaneous equations consisting of one quadratic and one linear equation, or of two quadratic equations in two unknown quantities.

The coefficients may be integers, numerical fractions, or algebraic monomials.

Graphical treatment is expected for equations of the types

$$x^2+y^2=a^2, \quad x^2-y^2=a^2, \quad xy=a, \quad y^2=ax.$$

Use

In addition to the above-named combinations with the linear and the quadratic equations in two unknown quantities, the solving of the calculus problems required: one linear and one cubic equation, one quadratic and one cubic equation, two cubic equations, and one linear and one radical equation.

The methods used to solve the equations listed in the syllabus could be used to solve the additional ones found in this study.

The treatment of the graphic solution for the specific types listed was not used; the solution of any two such equations was made by some other method. It was necessary to know the general shape of such a curve though it was not necessary to plot it in detail. Chapter V of this study lists all the standard curves used, both those expressed in rectangular coordinates and those expressed in polar coordinates.

6. EXPONENTS AND RADICALS

Requirements

(a) The theory and use of fractional, negative, and zero exponents.

(b) The rationalization of the denominator of such fractions as

$$\frac{a+\sqrt{b}}{c-\sqrt{d}}.$$

(c) The solution of such equations as

$$\sqrt{3-2x}-x=30.$$

Use

(a) Fractional, negative, and zero exponents were used to a great extent in the solution of the calculus problems. It was necessary to know the meaning of these terms before problems involving them could be solved. The ability to convert radical forms to equivalent forms with fractional exponents was also necessary.

(b) The rationalization of denominators was confined to fractions having monomial denominators, since it was the only case, except one fraction which had a binomial denominator, used in solving the calculus problems.

(c) The few cases of the solution of radical equations involved none more difficult than the form stated in the syllabus; that is, the solution could be effected by squaring each side of the equation but once, in order to clear it of radicals.

7. LOGARITHMS

Requirements

(a) The fundamental formulas.

(b) Computation by four-place tables.

(c) Application to the trigonometry of the right triangle.

Use

(a) The fundamental formulas of logarithms play an important part in the solution of the calculus problems. Understanding the theory of logarithms is important, for with such knowledge the student may set up many problems in such a way that their solution is much simpler than if they are left in the original form.

(b) Computation by logarithms is essential in the study of the calculus, but to a much less degree than the use of the fundamental formulas. One reason for this is that many answers can be left in the form of an indicated logarithm. The use of computation to base e was eight times as great as that to base *10*. In working with logarithms of either base, the ability to interpolate was necessary. Ability to convert from base e to base *10* was seldom needed; it can easily be taught when needed.

(c) The use of logarithms in the application of the trigonometry of the right triangle is listed under trigonometry.

With the exception of the application of the trigonometry of the right triangle, the uses of logarithms in the analyses made in this investigation included the requirements listed in the syllabus and a few additional facts. However, the extra knowledge needed for this topic can easily be taught when needed.

ADVANCED ALGEBRA

(A) Theory of Equations

Requirements

(a) The theorem that an equation of the nth degree has n roots, if every such equation has one root.

(b) Factoring a polynomial in one variable, and the remainder theorem.

(c) The coefficients as symmetric functions of the roots.

(d) Simple transformations of equations, limited to the removal of the second term, and increase of the roots by a given number and multiplication of the roots by a given factor.

(e) Conjugate complex roots of equations with real coefficients.

(f) Equations with whole numbers or fractions as coefficients. Conditions for a rational root.

(g) Approximate solution of numerical equations.

(h) Descartes' Rule of Signs.

(i) Preliminary location of roots by the graph.

(j) Determination of the roots to two or three significant figures.

Use

(a) For all equations solved, the knowledge that if there are any roots there are as many as the degree of the equation is essential. The use made of this fact is limited, however, to quadratic equations, cubic equations which for the most part could be solved by factoring, and biquadratic equations of the quadratic form. There were five equations in the study of degree greater than four.

(b) There were only eleven polynomials consisting of four or more terms that had to be factored; six were factored by grouping, and the others by the remainder theorem.

(c) The knowledge of the relationship of the coefficients of an equation to the roots is helpful in the study of the calculus, although it is not explicitly called for.

(d) There were no applications of the transformations listed in the requirements; however, if the irrational roots of equations were found by Horner's method, this knowledge would be necessary.

(e) In the solution of the quadratic equations involved in the solution of the calculus problems, there were a few whose roots were imaginary. This is the only use made of complex numbers.

(f) Little or no use was made of this fact.

(g) (h) There were thirty-nine equations to be solved graphically. The answers thus obtained are approximate. The knowledge of Descartes' Rule of Signs was of material aid in determining the number of positive and negative roots. Knowing the signs of the roots was an aid in selecting values for the independent variable in the table of values used in plotting.

(i) (j) There were seventeen equations whose roots were found by Newton's method.

The requirements listed under "Theory of Equations" are the common properties of equations and roots of equations. There were few explicit applications of these requirements in the solution of the calculus problems, but a knowledge of these requirements would give a better background for doing the work. However, in view of this limited use, most of the facts listed can be taught when needed if they have not been learned in elementary or intermediate algebra.

(B) Determinants

Requirements

Definition of determinants of the second and third orders by explicit polynomial formulas. Evaluation of such determinants by the familiar rules. The simple transformations, proved directly from the definition, and illustrated by examples in which the elements of the determinant are small integers.

Determinants of the fourth order; their evaluation and transformation.

Application to linear equations: (i) non-homogeneous equations in 2, 3, and 4 unknowns; (ii) homogeneous equations in 2 and 3 unknowns; the treatment to include all cases in which the number of equations does not exceed the number of unknowns. The case of compatibility of three non-homogeneous equations in two unknowns is also included.

Use

No use was made of determinants in solving equations that arose in the solution of the calculus problems.

(C) Brief Topics

Requirements

(a) Complex numbers, numerical and geometric treatment.
(b) Simultaneous quadratics.
(c) Scales of notation.
(d) Mathematical induction.
(e) Permutations and combinations. Probability.

Use

(a) The only examples of complex numbers used were the roots of certain quadratic equations, where the discriminant was less than zero. The student should be familiar with this concept, as imaginary roots appear in the solution of the quadratic equations in intermediate algebra.

(b) The only use of simultaneous equations in two unknowns was the use of the simple forms; these forms are discussed under simultaneous equations, on page 50.

(c) There are no references in the solution of the calculus problems to scales of notation other than base *10*.

(d) There was no direct application of mathematical induction to the problems in this study; however, it is useful in understand-

ing the theory relating to the sum of certain series and in expansion by the binomial theorem. In the latter there were only cases of positive integral exponents and in no case was the exponent greater than 3.

(e) Applications of permutations, combinations, and probability were not found among the problems of the calculus.

PLANE GEOMETRY

The requirements in plane geometry as stated in the syllabus of the College Entrance Examination Board consists of eighty-nine propositions and twenty construction problems. Certain propositions are starred to indicate that the student is required to know them; that is, any proposition from this group may be given in the examination. Propositions which are not so marked are not required in the examination, but the student is expected to be familiar with their content so that he may answer questions about their substance or use them as a basis for solving originals.

The list of the required propositions is not reproduced in detail in this summary; few of them were actually used in solving the calculus problems. Any proposition that was used in the solution of the calculus problems is stated, and likewise any corollary that was used. Those propositions which are starred in the syllabus are also starred with an asterisk in this summary.

The following constitute the propositions used in the solution of the calculus problems:

BOOK I

Triangles, Perpendiculars, and Parallels

(a) Triangles: Of the five propositions listed under this topic the only one used was

> * If two sides of a triangle are equal, the angles opposite these sides are equal.

(b) Parallels and Perpendiculars: Of the twelve propositions given under this heading none was used.

(c) Parallelograms: There are five propositions listed under parallelograms but the only one used was:

> The diagonals of a parallelogram bisect each other.

There were four other geometric facts used in this study that

were not listed in the syllabus, but they would follow naturally from the proposition and definitions of this topic. They are:

(1) In a right triangle the mid-point of the hypotenuse is equidistant from the three vertices.
(2) The diagonals of a rhombus are perpendicular to each other.
(3) The diagonals of a rhombus divide the rhombus into four congruent triangles.
(4) The formulas for the length of the perimeter of a rectangle and square.

(d) Sum of the Angles of a Triangle: Of the three propositions stated under this heading none was directly used, but the following geometric fact, which is based on the sum of the angles of a triangle, was needed.

In an isosceles right triangle each acute angle is 45°.

(e) Inequalities: Four propositions are given under this topic, but none was used.

(f) Loci: Of the two propositions given under loci both were starred but neither was directly used; one of them, however, is the basis for the following geometric fact that was used.

In an isosceles triangle the perpendicular from the vertical angle to the base bisects the base.

(g) Lines through a Point: The syllabus lists four propositions under this topic, but none was used in solving the problems of the calculus.

BOOK II

The Circle

(a) Circles in General: There are seven propositions under this heading, but none of them was used.

(b) Tangents: Of the four propositions listed, only one was used:

A line perpendicular to a radius at its outer extremity is tangent to the circle.

However, it was necessary to know the fact that

If two circles are tangent to the same line at the same point they are tangent to each other.

(c) The Measure of Angles: Six propositions are given under

this topic but there were no applications of them in the solution of the calculus problems.

(d) Inequalities: The four propositions here listed were not used at all.

BOOK III

Similar Polygons

(a) Similar Triangles: Of the five propositions stated under the heading "Similar Triangles" only the following was used:

> A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.

The two geometric facts that follow come under this topic and they were used in this study:

> Corresponding sides of similar triangles are in proportion.
>
> Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.

(b) Applications: Of the seven propositions listed under this heading the following two were used:

> (1) * In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.
>
> (2) In any right triangle the perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two triangles, each similar to the given triangle.

Proposition (2) above was not used directly, but the geometric fact that was used is that

> The perpendicular is the mean proportional between the segments of the hypotenuse.

Closely related to proposition (2) above is another geometric fact that was used in this study.

> The perpendicular from any point on a circle to its diameter is the mean proportional between the segments of the diameter.

(c) Similar Polygons: Of the three propositions that follow this topic none was used in the solution of the calculus problems. Under this topic were two geometric facts that were used:

> Finding the altitude of an equilateral triangle, and
>
> Finding the slant height of a cone.

Each is an application of the Pythagorean theorem.

BOOK IV

Areas of Polygons

Of the six propositions required in this division of plane geometry four were used in this study:

(1) * The area of a parallelogram is equal to the product of its base by its altitude: $A = bh$.

The application of this proposition was to rectangles and squares.

(2) The area of a triangle is equal to half the product of its base and its altitude: $A = \frac{1}{2}bh$.

A special case of this was finding the area of an equilateral triangle in terms of its sides.

(3) * The area of a trapezoid is equal to half the product of the sum of its bases by its altitude:

$$A = \tfrac{1}{2}(b+b')h.$$

(4) * The areas of two similar triangles are to each other as the squares of any two corresponding sides.

BOOK V

The Measure of the Circle

Of the twelve propositions required by the syllabus three were used in the solution of the calculus problems:

(1) * The area of a regular polygon is equal to half the product of its apothem by its perimeter.

(2) The circumference of a circle is equal to the product of its radius by twice the constant number π:

$$c = 2\pi r.$$

(3) The area of a circle is equal to half the product of its circumference by its radius; or to the product of the square of its radius by the constant number π.

$$A = \pi r^2.$$

Following the required propositions in the syllabus is a list of twenty construction problems. No construction work was necessary in order to solve the calculus problems; hence, the construction work involved in these problems had no direct application in this study.

SOLID GEOMETRY

BOOK VI

Lines and Planes in Space

(a) Lines and Planes in Space: Under this heading there are twenty-two propositions, but none was used.

(b) Dihedral Angles: There are seven propositions relating to dihedral angles in the requirements, but none was used in this study.

(c) Loci: Of the three propositions listed under this topic none had an application to the problems of the calculus.

(d) Polyhedral Angles: Of the four propositions in this division of Book VI none was directly used in solving the calculus problems.

BOOK VII

Prisms and Pyramids. Cylinders and Cones.

(a) Prisms and Parallelepipeds: Of the ten propositions listed here two were used:

> (1) * The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.

Under this proposition comes the corollary for

> The total surface of a rectangular solid and also specifically for a cube.
>
> (2) The volume of a parallelepiped is equal to the product of its base by its altitude.

The use made of this proposition was in reference to rectangular solids in general and to cubes in particular.

(b) Pyramids: There are six propositions listed under pyramids but none had a direct application in this study. However, it is to be remembered that the pyramid is the basis for the proof of the proposition concerning cones; these propositions were used in the solution of the calculus problems.

(c) Cylinders: Of the three propositions given under this heading two were used:

> (1) The lateral area of a circular cylinder is equal to the product of an element by the perimeter of a right section. For a cylinder of revolution:
>
> $$A = 2\pi rh.$$

(2) The volume of a circular cylinder is equal to the product of its base by its altitude.
For a cylinder of revolution:

$$V=\pi r^2 h.$$

(d) Cones: There were four propositions under this topic, but only two were used:

(1) The lateral area of a cone of revolution is equal to half the product of its slant height by the circumference of its base:

$$A=\pi rl.$$

(2) The volume of a circular cone is equal to one-third the product of its base by its altitude:

$$V=\tfrac{1}{3}\pi r^2 h.$$

Use was made of another proposition under cones, but it was not among the required ones in the syllabus. It is for the volume of a frustrum of a cone of revolution

$$V=\frac{1}{3}\pi h(R^2+r^2+Rr).$$

(e) Similar Solids: None of the four listed under similar solids had a direct application in the solution of the problems of the calculus.

BOOK VIII

The Sphere

(a) Of the first ten propositions given under the sphere in general none was used in this study.

(b) Spherical Triangles and Polygons: There are sixteen propositions included under this topic but none was used.

(c) The Measurement of the Sphere: Of the three propositions listed two were used:

(1) The area of a sphere is equal to the area of four great circles:

$$A=4\pi r^2.$$

(2) The volume of a sphere is equal to one-third the product of its radius by the area of its surface:

$$V=\tfrac{4}{3}\pi r^3.$$

Use was made of one proposition that was not included in the

required propositions relating to the sphere. It is for the surface of a zone of one base:

$$V=\frac{1}{3}\pi h^2(3r-h).$$

It might be well to add that where it was necessary to establish a relationship between the slant height and the radius and altitude of a cone in order to solve certain problems of the calculus, such a process has been classed as an application of the Pythagorean theorem, and not as an application of solid geometry.

TRIGONOMETRY

1

Requirements

Definitions of the six trigonometric functions of angles of any magnitude, as ratios. The computation of five of these ratios from any given one. Function of 0°, 30°, 45°, 60°, 90°, and angles differing from these by multiples of 90°.

Use

Understanding the meaning of the six trigonometric functions and understanding the relationship of one of these functions to the others were both essential in solving the calculus problems. Also, it was necessary to know the functions of certain specific angles, as those listed above. These fundamental requirements were used extensively in the solution of the calculus problems.

2

Requirements

Determination, by means of a diagram, of such functions as sin $(A-90°)$ in terms of trigonometric functions of A.

Use

The functions used most often were those of acute angles, but it was necessary to be able to find functions of obtuse angles as well.

3

Requirements

Circular measure of angles; length of an arc in terms of central angle in radians.

Use

It was essential to have the ability to deal with angles whose measure was expressed in radians as well as those expressed in degrees; and also to be able to convert angles expressed in degrees to radians and vice versa.

4

Requirements

Proofs of the following fundamental formulas and of simple identities derived from them.

(a) The Ratio Formulas:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x},$$
$$\csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

(b) The Pythagorean Formulas:

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

(c) The Addition Theorems:

$$\sin (x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos (x+y) = \cos x \cos y - \sin x \sin y,$$
$$\tan (x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

(d) The Double-Angle Formulas, for $\sin 2x$, $\cos 2x$, and $\tan 2x$.

Use

(a) All the Ratio Formulas listed in the requirements, and in addition $\tan x = \frac{1}{\cot x}$, which follows naturally from $\cot x = \frac{1}{\tan x}$, were used repeatedly.

(b) The three Pythagorean Formulas were used freely in the solution of the calculus problems.

(c) All the Addition Formulas listed were used. The $\sin (x-y)$ and the $\cos (x-y)$, and also $\sin x - \sin y$ were used. Of all the six cases of the sum and difference formulas none was used more than five times, while three of them were used but once each.

(d) All the Double-Angle Formulas listed were used, and also the Half-Angle Formulas, for $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$.

In addition to the fundamental formulas and identities listed under (a), (b), (c), and (d) above, it was necessary to know the

meaning of polar coordinates and hyperbolic functions. Each of the last named groups was used but three times.

The requirements as listed do not include all the formulas that were used in the solution of the calculus problems, but the few additional facts, except the sin $(x-y)$, the cos $(x-y)$, and the hyperbolic functions that were used were included in the Formulas of Trigonometry listed in the "Notes to Teachers." The use made of these formulas was not for their derivations, but rather as a substitute to change an expression to an equivalent form so that it could then be differentiated or integrated.

5

Requirements

> Solution of simple trigonometric equations of the general order of difficulty of the following:
> $6 \sin x + \cos x = 2$; $\cos 2x = \sin x$; $\tan (x+30°) = \cot x$.

Use

There was practically no case where it was necessary to solve trigonometric equations in this study. There was an extensive use of the functions in both differentiation and integration problems, but there were no equations of the above type where it was necessary to prove the equation to be an identity. The meaning of arc sin x, arc cos x, arc tan x, and arc cot x mentioned in the "Notes to Teachers," but not required in the syllabus, was necessary in order to solve some of the calculus problems.

6

Requirements

> Theory and use of logarithms, without the introduction of work involving infinite series. Use of trigonometric tables, with interpolation.

Use

Logarithms were used to solve certain calculus problems. The table of natural logarithms was used much more often than the table of common logarithms. Ability to use trigonometric tables, both for finding the function of an angle and for the inverse operation was necessary. In the use of the logarithmic tables and the trigonometric tables ability to interpolate was essential.

7

Requirements

Derivation of the Law of Sines and the Law of Cosines.

8

Requirements

Solution of right and oblique triangles with special reference to the applications.

Use of 7 and 8

The use of trigonometry throughout this study was not its direct application in solving problems, but rather as a contributing factor to the solution of the calculus problems. Both parts 7 and 8 above deal directly with the solution of problems involving triangular relationships, and they, being purely trigonometric methods, have no direct application in the solution of the calculus problems.

CHAPTER VII

SUMMARY AND CONCLUSIONS

THE facts and processes listed in this study were those used in the solution of 2,811 problems of the calculus. The total frequency was 29,971, which gave an average frequency of 10.6 per problem.

Of the total 276 different facts and processes used in this study 129 were classed as algebraic, 43 as geometric, 37 as trigonometric, and 67 as from the field of analytic geometry. These totals may appear small because of the fact that all definitions of mathematical terms were assumed to be known by the students; hence they were not included among the facts listed. The terms facts, principles, formulas, and operations are explained on page 3.

The field of mathematics, beginning with algebra and continuing through analytic geometry, includes the subjects that usually precede the study of the calculus. The knowledge acquired from the study of each of these prerequisite subjects has contributed toward the solution of the problems of the calculus, as shown in Chapter VI. In this summary the contributions from each of these subjects is considered separately, and certain inferences are drawn from the data presented in the various chapters.

The inclusion of any topic in the course of study for secondary mathematics can not be justified solely on the grounds that it is needed in the solution of the problems of the calculus; nor can the exclusion of certain topics be justified on the basis that they are not needed for the calculus. However, one criterion for the selection of curriculum material at any level is that of preparation for the work on a higher level. Granting that this criterion deserves some weight and that the material of the calculus is in itself worth while, the following conclusions seem justified.

1. *Elementary Algebra (Part I)*

All of the subject matter of algebra, up to quadratics, that is presented in the College Entrance Examination Board syllabus is used in solving the calculus problems. The prescribed course meets the needs of the student and helps him do successful work

in the calculus. If this knowledge is not acquired in his secondary mathematics courses in elementary algebra, then it will be necessary for the student to acquire it before he can understand the calculus.

2. *Elementary Algebra* (*Part II*)

The outline of the work required in elementary algebra for quadratics and beyond and the uses made of it in solving the calculus problems coincide almost as completely as do the requirements and uses in Part I. There is no topic in Part II of the syllabus that is not used, at least in part, but the requirements are somewhat fuller than the uses made of them demand. This is not objectionable, however, since every topic taught is used in the solution of the calculus problems.

3. *Advanced Algebra*

The topics included in the syllabus for advanced algebra are (*a*) theory of equations, (*b*) determinants, and (*c*) brief topics. Only part of the requirements of the theory of equations is used in the study of the calculus, their use being limited almost exclusively to quadratic, cubic, and biquadratic equations. The requirements for determinants contain nothing that was used in the solution of the problems. Of the five topics under the heading "Brief Topics" the only one used was the requirement relating to simultaneous quadratic equations.

Since the use made of advanced algebra in the solution of the calculus problems is so much less than the requirements set forth in the syllabus, the writer suggests that part of the time spent on advanced algebra might well be given to some of the other work that is actually used in the study of the calculus, specifically to analytic geometry.

4. *Plane and Solid Geometry*

Of the eighty-nine propositions required in the syllabus only twenty-two were used in solving the calculus problems; of the twenty required construction problems, none was used.

Of the ninety-two propositions in solid geometry listed in the syllabus, only ten were used in solving the calculus problems; these were the ones that can be expressed as formulas for the surfaces and volumes of solids.

Since it has been assumed that one of the main objectives of the

teaching of plane and solid geometry is preparation for the calculus, and since the uses of plane geometry in this study were limited to specific cases, and also since the formulas of solid geometry were used almost as often as those of plane geometry, it seems desirable to offer a year's work in a course of plane and solid geometry combined. In such a course the demonstration of proofs could be confined almost entirely to plane geometry. This arrangement would still give sufficient practice in formal proof of propositions, and would also include certain facts from solid geometry which could be treated informally.

5. *Trigonometry*

The operations of trigonometry used in solving the calculus problems do not include the solution of right and oblique triangles, or the solution of trigonometric equations to show that they are identities. Both of these are important topics in the study of trigonometry, and involve applications of the trigonometric facts. In this study the many substitutions of one function for another were necessary before the calculus problems could be solved.

Since it has been assumed that the study of trigonometry in the high school has as its aim preparation for higher mathematics, specifically the calculus, and since the syllabus contains much material that was not required in solving the calculus problems, it seems that some of the time now devoted to the study of trigonometry might better be given to the study of the parts of analytic geometry that are needed for the calculus.

6. *Analytic Geometry*

The fundamental formulas and equations from analytic geometry were used in the solution of the calculus problems. The ones that were used most frequently were the commonly known formulas and the equations of the well-known standard curves, expressed in both rectangular and polar coordinates. There was little or no need of the plotting of curves in detail, but it was necessary for the student to know the general shape of the curve so that he could draw the curve by inspection. Nearly all the curves are illustrated in the textbook on which this investigation was based; the necessary knowledge of the shape of the curve can therefore easily be acquired in the calculus class.

Since the method of plotting curves in rectangular coordinates is taught in secondary algebra classes, and the meaning of polar

coordinates is explained in the trigonometry courses in the high school, no great amount of new subject matter in analytic geometry is needed in the study of the calculus. A saving could be effected in the time that is now given to advanced algebra and trigonometry in the high school and the time thus saved devoted to the elementary notions of analytic geometry that are needed for the study of the calculus.

The writer suggests the following plan whereby the student will be able to acquire, in high school, the mathematics necessary for the study of the calculus. Beginning with the ninth grade the student will have four years for high school mathematics. Two of these years, probably the ninth and the eleventh, are sufficient for courses in elementary, intermediate, and advanced algebra. The tenth year may be devoted to a combined course in plane and solid geometry. During the twelfth year a course in mathematical analysis might be offered. This course would include, primarily, trigonometry, analytic geometry, and an introduction of the fundamental ideas of differential and integral calculus. This combined course would give the student a conception of the character and possibilities of modern mathematics and the relation of its several branches as parts of a unified whole, and at the same time would give the necessary preparation for college calculus.

Modifications, of course, may be made in the time given to the various subjects of secondary mathematics as set forth in the above plan. Other objectives will necessarily shift the emphasis, with the result that more time will be given to solid geometry or to trigonometry or to both and less time to analytic geometry and to the fundamentals of the calculus.

The writer offers the plan on the assumption that one of the main objectives of secondary mathematics is preparation for the calculus. This objective can be accomplished in the four years of high school if the facts, principles, and processes used in the solution of the calculus problems are made the basis for the courses in secondary mathematics.